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STABILITY OF THE UNIFORM ROTATION OF A GYROSTAT ROUND THE VERTICAL MAIN AXIS ON AN ABSOLUTELY SMOOTH HORIZONTAL PLANE*

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The motion of a gyrostat on an absolutely smooth plane is discussed. A Hamilton function which gives the canonical equations of motion is obtained. This admits of particular solutions, namely uniform rotations round a vertical axis which are identical with that of the uniform rotations of the rotor. A transition to a system with two degrees of freedom is realized, and the expansion of the Hamiltonian in the vicinity of the corresponding position of equilibrium, with an accuracy to within fourth-order terms, is obtained. In the region of admissible values of the parameters the domain of the necessary stability conditions, and the domains where the Hamiltonian functions are of fixed sign and alternating, are examined. In those cases where the Hamiltonian is not fixed sign, its normalization is performed, both a non-resonance situation and resonances of the first, second and fourth order being considered. The sufficient conditions for stability of uniform gyrostat rotation in terms of constraints on the coefficients of normal forms are obtained. For a clear interpretation of the results, special cases where the values of all the parameters except two are fixed, are given. The plane domain of the necessary stability conditions and resonance curves are constructed, and using computer results stability on the curves is discussed.

The stability of uniform rotations of a heavy solid around the vertical principal and minor axes on an absolutely smooth, and on an absolutely rough horizontal plane, and also on a plane with viscous friction is discussed in /1—4/. The stability of uniform rotations of a gyrostat round the vertical principal axis on absolutely smooth and absolutely rough horizontal planes was considered in /5, 6/. Investigations of the motion of a solid on an absolutely rough plane, the body being perturbed with respect to rotation round the principal axis (in particular with respect to the steady position of equilibrium), are described in

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/7, 8/. The stability of two types of rotation of a homogeneous ellipsoid on an absolutely smooth horizontal plane, in particular the stability of the uniform rotations of an ellipsoid round the vertical principal axis is discussed in /9/.

1. Consider the motion of a heavy solid under the influence of the force of gravity on an absolutely smooth horizontal plane. Suppose that the body (housing) has a cavity, and the axis of rotation of a symmetric gyroscope which rotates without friction in the cavity with a constant arbitrary angular velocity ω_2° relatively to the housing, is connected rigidly to the body. We will assume that the surface bounding the body is convex so that it comes into contact with the horizontal plane only at one of its points where the surface has a definite tangent plane. We introduce a fixed system of rectangular coordinates $OXYZ$, with the origin at the point O of the reference plane $Z = 0$, and the coordinate system $S\xi'\eta'\zeta'$ which is rigidly connected with the housing. The axes of the latter are directed along the principal centre axes of inertia of the gyrostat (i.e. of the housing-gyroscope system). We will assume that the axis of rotation of the gyroscope coincides with the axis $S\eta'$. We shall define the position of the housing by the coordinates X_s, Y_s of the point S, and by Euler's angles θ, φ and ψ which orient the system $S\xi'\eta'\zeta'$ with respect to $OXYZ$. The Hamilton function which defines the canonical equations of the gyrostat's motion has the form

$$H = \frac{1}{2\Delta}(\Phi(p_\theta - \alpha)^2 - 2\Psi(p_\theta - \alpha)(p_\varphi - \beta) + \Theta(p_\varphi - \beta)^2) - \tag{1.1}$$

$$\gamma + \frac{1}{2M}(p^2 + q^2)$$

$$\Delta = \Theta\Phi - \Psi^2, \ \Theta = I_{22} - I_{23}{}^2 I_{33}{}^{-1} + M\varkappa^2$$

$$\Phi = (I_{11} - I_{13}{}^2 I_{33}{}^{-1} + M\chi_2{}^2)\sin^2\theta$$

$$\Psi = (I_{12} - I_{13} I_{22} I_{33}{}^{-1} + M\varkappa\chi_2)\sin\theta$$

$$\alpha = \Lambda I_{23} I_{33}^{-1} - D\omega_2^\circ \sin\varphi, \quad \beta = \Lambda(I_{33}\cos\theta + I_{13}\sin\theta) I_{33}^{-1}$$

$$\gamma = -\frac{1}{2}\Lambda^2 I_{33}^{-1} + Mg(\chi_1 \sin\theta + \zeta'\cos\theta)\,\Lambda =$$

$$p_\psi - D\omega_2^\circ \sin\theta\cos\varphi, \quad \varkappa = \chi_1\cos\theta - \zeta'\sin\theta\,\chi_1 =$$

$$\xi'\sin\varphi + \eta'\cos\varphi, \quad \chi_2 = \xi'\cos\varphi - \eta'\sin\varphi$$

$$I_{11} = (A\sin^2\varphi + B\cos^2\varphi)\cos^2\theta + C\sin^2\theta, \ I_{22} =$$

$$A\cos^2\varphi + B\sin^2\varphi, \ I_{33} = (A\sin^2\varphi + B\cos^2\varphi)\sin^2\theta +$$

$$C\cos^2\theta, \ I_{12} = -(A - B)\sin\varphi\cos\varphi\cos\theta, \ I_{13} = -(A\sin^2\varphi +$$

$$B\cos^2\varphi - C)\sin\theta\cos\theta, \ I_{23} = (A - B)\sin\varphi\cos\varphi\sin\theta)$$

Here $p, q, p_\theta, p_\varphi, p_\psi$ are the gneralized momenta which correspond to $X_s, Y_s, \theta, \varphi, \psi$; M is the mass of the gyrostat; g is the acceleration due to gravity; ξ', η', ζ' are the coordinates of the point of contact between the body and the plane in the system $S\xi'\eta'\zeta'$, which are functions of θ and φ determined by the form of the equation defining the housing surface, and at the same time

$$(\xi^{\cdot\prime}\sin\varphi + \eta^{\cdot\prime}\cos\varphi)\sin\theta + \zeta^{\cdot\prime}\cos\theta \equiv 0$$

where the point denotes differentiation with respect to θ or φ; $I_{ij}\ (i, j = 1, 2, 3)$ are the components of the energy tensor of the gyrostat relative to the right-hand orthogonal system of coordinates $SX'Y'Z'$ whose SZ' axis is directed vertically upward, the SY' axis runs on the line of nodes in the direction in which SZ' axis rotates anticlockwise by an angle θ until it coincides with the $S\zeta'$ axis; A, B, C are the principal central moments of inertia of the gyrostat, and D denotes its axial moment of inertia.

2. The canonical equations of the gyrostat motion with the Hamilton function (1.1) admit of the particular solution

$$p = p_0, \ q = q_0, \ p_\theta = p_\varphi = 0 \tag{2.1}$$

$$p_\psi = B\omega_1^\circ + D\omega_2^\circ, \ X_s = M^{-1}p_0 t + X_s^\circ$$

$$Y_s = M^{-1}q_0 t + Y_s^\circ, \ \theta = \pi/2, \ \varphi = 0, \ \psi = \omega_1^\circ t + \psi_0$$

which corresponds to uniform rotation of the housing, with an arbitrary angular velocity ω_1° around the $S\eta'$ axis which is vertical. Here the centre of mass S of the gyrostat moves with a constnat velocity along a horizontal straight line. Without loss of generality we can assume that the centre is fixed. The coordinates X_s, Y_s and ψ are cyclic, and therefore the system discussed has two degrees of freedom.

We consider the perturbations

$$p_\theta = x_1', \quad p_\varphi = x_2', \quad \theta = \pi/2 + y_1', \quad \varphi = y_2'$$

and find an expansion of the Hamilton function of the system in the vicinity of the position of equilibrium, which corresponds to the stable motion (2.1), with an accuracy to within fourth-order terms with respect to x_1', x_2', y_1' and y_2'. Let h be the distance from the centre of mass S to the point of contact of the gyrostat and the plane during the unperturbed motion (2.1), $\nu' = h + \eta'$. Let us introduce new dimensionless variables x_1, x_2, y_1 and y_2, time τ, the dimensionless coordinates ξ, ν and ζ, the angular velocities ω_1 and ω_2 and the parameters a, b and n, using the formulae

$$x_i' = (BMgh)^{1/2} x_i, \quad y_i' = y_i, \quad i = 1, 2; \quad \tau = \left(\frac{Mgh}{B}\right)^{1/2} t$$

$$\xi = \frac{\xi'}{h}, \quad \nu = \frac{\nu'}{h} = 1 + \frac{\eta'}{h} = 1 + \eta, \quad \zeta = \frac{\zeta'}{h}$$

$$\omega_1 = \left(\frac{B}{Mgh}\right)^{1/2} \omega_1^\circ, \quad \omega_2 = \frac{D}{(BMgh)^{1/2}} \omega_2^\circ$$

$$a = \frac{B}{A}, \quad b = \frac{B}{C}, \quad n = \frac{Mh^2}{A}$$

Then,

$$\begin{aligned}
2H = {} & ax_1^2 + 2\Omega x_1 y_2 + bx_2^2 + 2\omega_1 x_2 y_1 + \omega_1(\omega_1 + \omega_2) y_1^2 + \\
& \Omega(\omega_1 + \omega_2) y_2^2 - anx_1^2 y_1^2 - (a-1) x_1^2 y_2^2 + \\
& 2(a - 1 - bn) x_1 x_2 y_1 y_2 + (\Omega - \omega_2 - 2a(\omega_1 + \omega_2) n) x_1 y_1^2 y_2 - \\
& \left(\frac{4}{3}(a-1)\omega_1 + \left(\frac{4}{3}a - 1\right)\omega_2\right) x_1 y_2^3 + x_2^2 y_1^2 - \frac{b^2}{a} n x_2^2 y_2^2 + \\
& \left(\frac{5}{3}\omega_1 + \omega_2\right) x_2 y_1^3 + (2\Omega - \omega_2 - 2b(\omega_1 + \omega_2) n) x_2 y_1 y_2^2 + \\
& \left(\frac{2}{3}\omega_1^2 + \frac{1}{4}\omega_2\left(\frac{11}{3}\omega_1 + \omega_2\right)\right) y_1^4 + h_{22} y_1^2 y_2^2 + h_{04} y_2^4 + \\
& n\left(ax_1^2 + 2\Omega x_1 y_2 + \frac{1}{a}\Omega^2 y_2^2\right)(2\zeta y_1 - \zeta^2) + \\
& 2n\left(bx_1 x_2 + \frac{b}{a}\Omega x_2 y_2 + \omega_1 x_1 y_1 + \frac{1}{a}\omega_1 \Omega y_1 y_2\right)(-\xi y_1 + \zeta y_2 + \xi\zeta) - \\
& \frac{b}{a} n\left(bx_2^2 + 2\omega_1 x_2 y_1 + \frac{\omega_1^2}{b} y_1^2\right)(2\xi y_2 + \xi^2) + \\
& \left(-2y_2 + y_1^2 y_2 + \frac{y_2^3}{3}\right)\xi - y_1^2 - y_2^2 + \frac{y_1^4}{12} + \frac{y_1^2 y_2^2}{2} + \\
& \frac{y_2^4}{12} + (-2 + y_1^2 + y_2^2)\nu + \left(2y_1 - \frac{y_1^3}{3}\right)\zeta
\end{aligned} \tag{2.2}$$

Here

$$\begin{aligned}
\Omega = {} & (a-1)\omega_1 + a\omega_2, \quad h_{22} = \frac{1}{a}(a-1)\Omega(\omega_1 + \omega_2) + \\
& \frac{b}{a}(2\Omega - a\omega_2)\omega_1 + \left(1 - \frac{1}{a}\right)\left(1 - \frac{2}{b}\right)\omega_1^2 + \\
& \left(\frac{3}{2} - \frac{1}{a} - \frac{1}{b}\right)\omega_1\omega_2 + \frac{1}{2}\omega_2^2 - \frac{1}{a}(\omega_1^2 + \Omega(\Omega - 2\omega_1)) n \\
h_{04} = {} & -\left(1 - \frac{1}{a}\right)\left(\frac{1}{3} + \frac{1}{a}\right)\Omega\omega_1 - \frac{1}{3}\Omega\omega_2 + \\
& \frac{1}{4}\left(\left(\frac{11}{3} - \frac{4}{a}\right)\omega_1 + \omega_2\right)\omega_2 + \left(1 - \frac{1}{a}\right)\left(\frac{2}{3} - \frac{1}{a}\right)\omega_1^2
\end{aligned}$$

We shall investigate the stability of the uniform rotations (2.1) of the gyrostat with respect to p_θ, p_φ, θ, φ, p_ψ for parametric perturbations of its constructional parameters (see /10/).

3. We shall henceforth assume that in a small vicinity of the contact point between the gyrostat and the plane during the steady motion (2.1), the surface of the housing defined by the equation

$$f(\xi', \eta', \zeta') = 0 \quad (f(0, -h, 0) = 0) \tag{3.1}$$

is close to an ellipsoid, one axis of which lie on the $S\eta'$ axis so that

$$f(\xi', \eta', \zeta') = -\eta' - h + \frac{1}{2}(P\xi'^2 - 2Q\xi'\zeta' + R\zeta'^2) + \frac{1}{8h}(P\xi'^2 - 2Q\xi'\zeta' + R\zeta'^2)^2 \tag{3.2}$$

$$P = \frac{\cos^2\varepsilon}{r_1} + \frac{\sin^2\varepsilon}{r_2}, \quad Q = \left(\frac{1}{r_1} - \frac{1}{r_2}\right)\sin\varepsilon\cos\varepsilon$$

$$R = \frac{\sin^2\varepsilon}{r_1} + \frac{\cos^2\varepsilon}{r_2}$$

Here r_1 and r_2 are the principal radii of curvature of the surface (3.1) at the point $(0, -h, 0)$, ε is the angle between the $S\zeta'$ axis and the direction of principal curvature corresponding to r_2, which is measured anticlockwise from the $S\zeta'$ axis looking towards the $S\eta'$ axis directed vertically upwards during the unperturbed motion (2.1).

Let us introduce the dimensionless quantities

$$l = \frac{r_1 r_2 Q}{h}, \quad l_1 = \frac{r_1 r_2 P}{h}, \quad l_2 = \frac{r_1 r_2 R}{h}$$

Then, considering (3.2), we obtain the following expressions for the dimensionless coordinates ξ, η, ζ by means of the dimensionless perturbations y_1, y_2 with accuracy to within fourth-order terms with respect to the perturbations

$$\begin{aligned}
\xi &= ly_1 - l_2y_2 + \frac{1}{2}\Big\{-l\Big(l_1 - \frac{2}{3}\Big)y_1^3 + \\
&(l_1l_2 + 2l^2)y_1^2y_2 - l(3l_2 - 1)y_1y_2^2 + l_2\Big(l_2 - \frac{2}{3}\Big)y_2^3\Big\} \\
\eta &= -1 + \frac{1}{2}(l_1y_1^2 - 2ly_1y_2 + l_2y_2^2) + \frac{1}{8}\Big\{-l_1\Big(3l_1 - \frac{8}{3}\Big)y_1^4 + \\
&l\Big(12l_1 - \frac{8}{3}\Big)y_1^3y_2 - (6(l_1l_2 + 2l^2) - 4l_1)y_1^2y_2^2 + \\
&l\Big(12l_2 - \frac{20}{3}\Big)y_1y_2^3 - l_2\Big(3l_2 - \frac{8}{3}\Big)y_2^4\Big\} \\
\zeta &= l_1y_1 - ly_2 + \frac{1}{2}\Big\{-l_1\Big(l_1 - \frac{2}{3}\Big)y_1^3 + 3ll_1y_1^2y_2 - \\
&(l_1l_2 + 2l^2 - l_1)y_1y_2^2 + l\Big(l_2 - \frac{2}{3}\Big)y_2^3\Big\}
\end{aligned} \tag{3.3}$$

On substituting formulae (3.3) into (2.2), we obtain the final expansion of the Hamilton functions of our system, with an accuracy to within fourth-order terms,

$$H = H_2 + H_4 + \ldots \tag{3.4}$$
$$H_k = \sum_{|\nu|=k} h_{\nu_1\nu_2\nu_3\nu_4} x_1^{\nu_1}x_2^{\nu_2}y_1^{\nu_3}y_2^{\nu_4}$$

where $\nu_1, \nu_2, \nu_3, \nu_4$ are non-negative integers, and $|\nu| = \nu_1 + \nu_2 + \nu_3 + \nu_4$, $k = 2, 4$. The non-zero coefficients $h_{\nu_1\nu_2\nu_3\nu_4}$ have the form

$$\begin{aligned}
&2h_{2000} = a,\ h_{1001} = \Omega,\ 2h_{0200} = b,\ h_{01\,)} = \omega \\
&2h_{0020} = \omega_1\ (\omega_1 + \omega_2) - (1 - l_1),\quad 2h_{0002} = \Omega\ (\omega_1 + \omega_2) - \\
&\quad (1 - l_2) \\
&h_{0011} = -l,\ 2h_{2020} = an\ (l_1\ (2 - l_1) - 1) \\
&h_{2011} = -anl\ (1 - l_1),\ 2h_{2002} = -(a - 1) - anl^2 \\
&h_{1120} = -bnl\ (1 - l_1),\ h_{1111} = a - 1 - bn\ (1 + l^2 + l_1l_2 - \\
&\quad l_1 - l_2),\ h_{1102} = -bnl\ (1 - l_2),\ h_{1030} = -\omega_1nl\ (1 - l_1) \\
&2h_{1021} = \Omega - \omega_2 - 2(a(\omega_1 + \omega_2) - \Omega l_1(2 - l_1) + \\
&\quad \omega_1(l^2 + l_1l_2 - l_1 - l_2))n,\quad h_{1012} = -(2\Omega l(1 - l_1) + \\
&\quad \omega_1l(1 - l_2))n,\quad 2h_{1003} = -\Big(\frac{4}{3}\Omega - \omega_2 + 2\Omega l^2n\Big) \\
&2h_{0220} = 1 - \frac{b^2}{a}nl^2,\quad h_{0211} = -\frac{b^2}{a}nl(1 - l_2) \\
&2h_{0202} = -\frac{b^2}{a}n(1 - l_2(2 - l_2)),\quad 2h_{0130} = \frac{5}{3}\omega_1 + \omega_2 - \\
&\quad 2\frac{b}{a}\omega_1nl^2,\quad h_{0121} = -\frac{b}{a}nl(\Omega(1 - l_1) + 2\omega_1(1 - l_2)) \\
&2h_{0112} = 2\Omega - \omega_2 - 2b(\omega_1 + \omega_2)n - 2\frac{b}{a}(\Omega(l^2 + l_1l_2 - l_1 - l_2) - \\
&\quad \omega_1l_2(2 - l_2))n,\quad h_{0103} = -\frac{b}{a}\Omega nl(1 - l_2) \\
&2h_{0040} = \frac{2}{3}\omega_1^2 + \frac{1}{4}\omega_2\Big(\frac{11}{3}\omega_1 + \omega_2\Big) - \frac{1}{a}\omega_1^2nl^2 - \frac{l_1^2}{4} + \\
&\quad \frac{l_1}{6} + \frac{1}{12},\quad 2h_{0031} = -\frac{2}{a}\omega_1nl(\Omega(1 - l_1) + \omega_1(1 - l_2)) + \\
&\quad l\Big(l_1 + \frac{1}{3}\Big),\quad 2h_{0022} = h_{22} + \frac{1}{a}\Omega(\Omega l_1(2 - l_1) - \\
&\quad 2\omega_1(l^2 + l_1l_2 - l_1 - l_2))n + \frac{1}{a}\omega_1^2nl_2(2 - l_2) - \\
&\quad \frac{1}{2}(2l^2 + l_1l_2) + \frac{1}{2}(l_1 - l_2 + 1),\quad 2h_{0013} = \\
&\quad -\frac{2}{a}\Omega l(\Omega(1 - l_1) + \omega_1(1 - l_2))n + l\Big(l_2 - \frac{2}{3}\Big) \\
&2h_{0004} = h_{04} - \frac{1}{a}\Omega^2l^2n - \frac{1}{4}l_2^2 + \frac{l_2}{6} + \frac{1}{12}
\end{aligned} \tag{3.5}$$

Note 3.1. The coefficients of the form H_2 depend on seven constructional parameters $\mathbf{c} = (\omega_1, \omega_2, a, b, l, l_1, l_2)$, and those of the form H_4 depend additionally on the parameter n.

Note 3.2. In /9/ an expansion of the Hamilton function of the system in question was obtained in the vicinity of the position of equilibrium which corresponds to the uniform rotation of a homogeneous ellipsoid about the vertical, accurate to fourth-order terms. The coefficients of this expansion, which depend on three dimensionless parameters k, δ_1 and δ_2, are easy to obtain from (3.5). In fact, if we impose some constraints on the gyrostat, the parameters $\mathbf{c}$ and n will be connected with the above parameters by the expressions

$$\omega_1^2=\omega^2=\frac{\delta_1+\delta_2}{5k},\quad \omega_2=0,\quad a=\frac{\delta_1+\delta_2}{1+\delta_2}$$
$$b=\frac{\delta_1+\delta_2}{1+\delta_1},\quad l=0,\quad l_1=\delta_2,\quad l_2=\delta_1,\quad n=\frac{5}{1+\delta_2}$$

Let us substitute them into (3.5). We allow for the fact that the dimensionless variable and time in /9/ were introduced by formulae different from those in Sect.2, and this corresponds to multiplying the coefficients $h_{\nu_1\nu_2\nu_3\nu_4}$, containing ω^m, $m=1,2$ by the quantity $(5k/(\delta_1+\delta_2))^{m/2}$. Then we obtain what is required.

4. Consider the domain of admissible values of the parameters

$$F=\{\mathbf{c}: a<b\,(a+1),\ b<a\,(b+1),\ a>b\,(a-1),\ l_1>0,\ \ l_2>0\}$$

and the domain of the necessary stability conditions of the solutions of (2.1),

$$G=\{\mathbf{c}:\mathbf{c}\in F,\ Q_1>0,\ Q_2>0,\ Q_1^2-4Q_2>0\}$$

In the domain G we shall examine the domain G_1 of positive definiteness and the region G_2 of sign alteration of the quadratic form H_2 of the Hamiltonian

$$G_1=\{\mathbf{c}:\mathbf{c}\in G,\ \lambda>0\},\ G_2=\{\mathbf{c}:\mathbf{c}\in G,\ \lambda<0\} \tag{4.1}$$

Here

$$\lambda=((b-1)\,\omega_1+b\omega_2)\,\omega_1-b\,(1-l_1) \tag{4.2}$$
$$Q_1=\omega_1^2+\Omega\,((b-1)\ \omega_1+b\omega_2)-a\,(1-l_1)-b\,(1-l_2)$$
$$Q_2=(\Omega\omega_1-a\,(1-l_2))\,\lambda-abl^2$$

The frequencies of a system with the Hamiltonian H_2 are

$$\alpha_{1,2}=\frac{1}{\sqrt{2}}(Q_1\pm\sqrt{Q_1^2-4Q_2})^{1/2} \tag{4.3}$$

Further we shall need to consider the resonance hypersurfaces of the first, second and fourth order,

$$R_1=\{\mathbf{c}:\mathbf{c}\in F,\ \alpha_2(\mathbf{c})=0\} \tag{4.4}$$
$$R_{N+1}=\{\mathbf{c}:\mathbf{c}\in F,\ \alpha_1(\mathbf{c})=N\alpha_2(\mathbf{c})\},\ N=1,3$$

It will be shown below that all the domains and hypersurfaces indicated are non-empty, R_1 and R_2 defining in F the boundary of the domain G and $G_2\cap R_4\neq\varnothing$.

Let $\mathbf{c}\in G_1$. In accordance with the Routh theorem complemented by Lyapunov (see, for example, /11/) the uniform rotations (2.1) are stable. To examine stability in the cases where $\mathbf{c}\in\partial G_1,\ \mathbf{c}\in G_2\setminus R_4,\ \mathbf{c}\in G_2\cap R_4,\ \mathbf{c}\in\partial G_2$ one must normalize the Hamilton function.

Note 4.1. In studying the stability of the uniform rotations of a heavy solid and a gyrostat with a fixed point round the principal vertical axis, it was established in /12–14/ that in this problem the function Q_2 may be presented in the form of a product of two Poincaré stability coefficients. Consequently, the hypersurface dividing the domains of sign alternation and of fixed sign of the Hamiltonian is at the same time a boundary of domain G. In our problem, this only holds for $l=0$.

Note 4.2. The problem discussed differs from those in /12–14/ by a large number of constructional parameters, and therefore a detailed analysis of domain G can be performed in special cases only.

Note 4.3. The quantities Q_1 and Q_2 are identical with the corresponding quantities in /6/ to within a positive multiplier and apart from notation.

5. Let us reduce the Hamilton function to normal form. We introduce the following notation:

$$f_1(\alpha_1)=\alpha_1^2-\Omega\,((b-1)\,\omega_1+b\omega_2)+b\,(1-l_2) \tag{5.1}$$
$$f_2(\alpha_1)=-\Omega\alpha_1^2+\omega_1^2\Omega-a\omega_1\,(1-l_2)$$
$$f_3(\alpha_1)=a\alpha_1^2-b\,(\omega_1\Omega-a\,(1-l_2))$$

To carry out the linear normalization we make the following change:

$$\begin{Vmatrix}x_1\\x_2\\y_1\\y_2\end{Vmatrix}=\begin{Vmatrix}s_1&c_1&t_1&d_1\\s_2&c_2&t_2&d_2\\s_3&c_3&t_3&d_3\\s_4&c_4&t_4&d_4\end{Vmatrix}\ \begin{Vmatrix}p_1\\p_2\\q_1\\q_2\end{Vmatrix} \tag{5.2}$$

Let $\mathbf{c}\in G_2$. Then

$$s_1 = \alpha_1 f_1(\alpha_1)\, g(\alpha_1),\ s_2 = -\alpha_1 alg(\alpha_1)$$
$$s_3 = 0,\ s_4 = \alpha_1 (b\,\Omega - a\omega_1)\, g(\alpha_1),\ t_1 = bl\Omega g(\alpha_1)$$
$$t_2 = f_2(\alpha_1)\, g(\alpha_1),\ t_3 = f_3(\alpha_1)\, g(\alpha_1)$$
$$t_4 = -ablg(\alpha_1),\ \ \alpha_1 g^2(\alpha_1) = \delta\, [f_1(\alpha_1) f_3(\alpha_1) + a^2bl^2 - f_2(\alpha_1)\ \ (b\Omega - a\omega_1)]^{-1} \tag{5.3}$$

where δ is the rank of the canonical transformation of (5.2). We can take $\delta = 1$ or $\delta = -1$ from the condition for the transformation to be real. The formulae for c_i, d_i $(i = 1, 2, 3, 4)$ are found from the expressions for s_i and t_i by replacing α_1 by α_2 respectively. Substitution of formulae (4.2), (4.3), (5.1) and (5.3) yields the final expression through the initial parameters of the problem for the coefficient of linear canonical transformation (5.2).

We write the fourth-order terms in the expansion (3.4) in the variables p_1, p_2, q_1, q_2 in the form

$$\delta K_4 = \sum_{|\nu|=4} \delta g_{\nu_1\nu_2\nu_3\nu_4} p_1^{\nu_1} p_2^{\nu_2} q_1^{\nu_3} q_2^{\nu_4} \tag{5.4}$$

and find the coefficients $g_{\nu_1\nu_2\nu_3\nu_4}$ required to study the stability

$$\delta g_{4000} = \sum_{|\nu|=4} h_{\nu_1\nu_2\nu_3\nu_4} s_1^{\nu_1} s_2^{\nu_2} s_3^{\nu_3} s_4^{\nu_4} \tag{5.5}$$

The coefficient g_{0400} is obtained from g_{4000} by replacing the quantities s_i by c_i, g_{0040} by replacing s_i by t_i, and g_{0004} by replacing s_i by d_i $(i = 1, 2, 3, 4)$

$$\delta g_{2200} = \sum_{|\nu|=4} (s_{\mu_1}s_{\mu_2}c_{\mu_3}c_{\mu_4} + c_{\mu_1}c_{\mu_2}s_{\mu_3}s_{\mu_4} + s_{\mu_1}c_{\mu_2}s_{\mu_3}c_{\mu_4} + c_{\mu_1}s_{\mu_2}c_{\mu_3}s_{\mu_4} + s_{\mu_1}c_{\mu_2}c_{\mu_3}s_{\mu_4} + c_{\mu_1}s_{\mu_2}s_{\mu_3}c_{\mu_4})\, h_{\nu_1\nu_2\nu_3\nu_4} \tag{5.6}$$

Here and below the quantities $\mu_1, \mu_2, \mu_3, \mu_4$ are computed from $\nu_1, \nu_2, \nu_3, \nu_4$ using the rows of the table. To obtain the remaining coefficients with non-zero subscripts 2 and 2 it is

Table 1

ν_l	ν_k	ν_j	ν_i	μ_1	μ_2	μ_3	μ_4
4	0	0	0	l	l	l	l
3	1	0	0	l	l	l	k
2	2	0	0	l	l	k	k
2	1	1	0	l	l	k	j
1	1	1	1	l	k	j	i

necessary to make the following substitutions on the right-hand side of formula (5.6):

$$g_{2020}: c_i \to t_i;\ g_{2002}: c_i \to d_i;\ g_{0220}: s_i \to t_i;\ g_{0202}: s_i \to d_i;$$
$$g_{0022}: s_i \to t_i;\ c_i \to d_i\ (i = 1, 2, 3, 4)$$
$$\delta g_{1300} = \sum_{|\nu|=4} (s_{\mu_1}c_{\mu_2}c_{\mu_3}c_{\mu_4} + c_{\mu_1}s_{\mu_2}c_{\mu_3}c_{\mu_4} + c_{\mu_1}c_{\mu_2}s_{\mu_3}c_{\mu_4} + c_{\mu_1}c_{\mu_2}c_{\mu_3}s_{\mu_4})\, h_{\nu_1\nu_2\nu_3\nu_4}$$

With respect to (5.7)

$$g_{1003}: c_i \to d_i;\ g_{0310}: s_i \to t_i$$
$$g_{0013}: s_i \to t_i,\ c_i \to d_i\ (i = 1, 2, 3, 4)$$

$$\delta g_{1201} = \sum_{|\nu|=4} s_{\mu_1}d_{\mu_2}c_{\mu_3}c_{\mu_4} + d_{\mu_1}s_{\mu_2}c_{\mu_3}c_{\mu_4} + s_{\mu_1}c_{\mu_2}d_{\mu_3}c_{\mu_4} + d_{\mu_1}c_{\mu_2}s_{\mu_3}c_{\mu_4} + s_{\mu_1}c_{\mu_2}c_{\mu_3}d_{\mu_4} + d_{\mu_1}c_{\mu_2}c_{\mu_3}s_{\mu_4} + c_{\mu_1}s_{\mu_2}d_{\mu_3}c_{\mu_4} + c_{\mu_1}d_{\mu_2}s_{\mu_3}c_{\mu_4} + c_{\mu_1}s_{\mu_2}c_{\mu_3}d_{\mu_4} + c_{\mu_1}d_{\mu_2}c_{\mu_3}s_{\mu_4} + c_{\mu_1}c_{\mu_2}s_{\mu_3}d_{\mu_4} + c_{\mu_1}c_{\mu_2}d_{\mu_3}s_{\mu_4})\, h_{\nu_1\nu_2\nu_3\nu_4} \tag{5.8}$$

With respect to (5.8)

$$g_{1102}: c_i \to d_i,\ d_i \to c_i;\ g_{0211}: s_i \to t_i$$
$$g_{0112}: s_i \to c_i,\ c_i \to d_i,\ d_i \to t_i\ (i = 1, 2, 3, 4)$$

Let $c \in G_2 \setminus R_4$. Then the coefficients of the normal form (3.4) are

$$2c_{20} = 3g_{4000} + 3g_{0040} + g_{2020},\ 2C_{02} = 3g_{0400} + 3g_{0004} + g_{0202},\ c_{11} = g_{2200} + g_{2002} + g_{0220} + g_{0022} \tag{5.9}$$

(see /15, 16/).

Substituting (3.5), (4.2), (4.3), (5.1), (5.3), (5.5) and (5.6) into (5.9) we obtain the final form of the coefficients. If $D^\circ \equiv c_{20}\alpha_2^2 + c_{11}\alpha_1\alpha_2 + c_{02}\alpha_1^2 \neq 0$, then by the Arnold-Moser theorem and its extension to stable motion, the uniform rotations (2.1) are stable (see /16, 15, 12/).

Let $c \in G_2 \cap R_4$. The coefficient of the resonance term is written in the form

$$C_4 = (x_{1003}^2 + y_{1003}^2)^{1/2},\ 2x_{1003} = g_{1300} + g_{0013} - g_{0211} - g_{1102},\ 2y_{1003} = -g_{0310} + g_{1003} - g_{1201} + g_{0112} \tag{5.10}$$

Substituting formulae (3.5), (4.2), (4.3), (5.1), (5.3) and (5.5)–(5.9) into (5.10) we obtain the final form of coefficient C_4. If $|D^\circ|/\alpha_2^2 - 3\sqrt{3}\,C_4 > 0$, then in accordance with Markeyev's theorem, /17/, and its extension to steady motions, /12/, the uniform rotations (2.1) are stable. If $D^\circ|/\alpha_2^2 - 3\sqrt{3}\,C_4 < 0$, the unperturbed motions (3.1) are unstable (see /17/).

Let $c \in \partial G \cap R_1$. This means that $c \in \partial G_1$, or that c belongs to that boundary of the domain G_2 which is defined by a first-order resonance relation. The coefficients of transformation (5.2) necessary to study stability have the form

$$c_1 = bl\Omega g_1,\; c_2 = f_2(0)\,g_1,\; c_3 = f_3(0)\,g_1,\; c_4 = -ablg_1,\; g_1^2 = \delta\,[f_1(0)\,f_3(0) + a^2bl^2 - f_2(0)\,(b\Omega - a\omega_1)]^{-1} \tag{5.11}$$

The fourth-order terms in the new canonical variables in expansion (3.4) are also written in the form (5.4), where the coefficient g_{0400}, necessary for the study, is given by Eq.(5.5). Subsituting (3.5), and (5.1) into (5.5) when $\alpha_1 = 0$ and (5.11) we obtain the final form of the coefficient g_{0400}. If $g_{0400} > 0$, the uniform rotations (2.1) are stable for fixed values of the parameters (see /18, 19/). If $g_{0400} < 0$, the solutions (2.1) are unstable /18, 19/.

Let $c \in \partial G_2 \cap R_2$. The coefficients required to investigate s_i, c_i $(i = 1, 2, 3, 4)$ of transformation (5.2) are computed from (5.3), in which the quantities t_i should be replaced by c_i, and $g(\alpha_1)$ by g_2. Quite cumbersome operations show that g_2 is chosen from the condition

$$2\alpha_1^2 g_2^2 = \delta\,[f_3(\alpha_1)]^{-1} \tag{5.12}$$

Substitution of formulae (4.2), $\alpha_1 = (Q_1/2)^{1/2}$, (5.1) and (5.12) into (5.3) results in the final expressions for the coefficients s_i and c_i $(i = 1, 2, 3, 4)$.

Using the new canonical variables, we shall write the fourth-order terms of (3.4) in the form (5.4), where the coefficients necessary to study the stability, g_{4000}, g_{0400} and g_{2200} are found from formulae (5.5) and (5.6). Let us put

$$E = 3g_{4000} + 3g_{0400} + g_{2200} \tag{5.13}$$

On substituting formulae (3.5), (4.2), $\alpha_1 = (Q_1/2)^{1/2}$, (5.1), (5.12), (5.3), (5.5) and (5.6) into (5.13), we obtain the final form of the coefficient E. If $E > 0$, the uniform rotations (2.1) are stable (see /20, 21/), and for $E < 0$ the steady rotations (2.1) are unstable (see /17/).

Note 5.1. In cases where $c \in \partial G$, the determining matrix of the system with a Hamiltonian H_2 has non-simple elementary divisors.

Note 5.2. In the problems discussed in /12–14/, the number of parameters on which the coefficients of the form H_k depend, are identical for H_2 and H_4. In our case, the coefficients H_4 depend on an additional parameter n. This means that the uniform rotations of a gyrostat on a plane have the following property. If $c \in G_1$, then the stability of rotations of the gyrostats whose parameters are represented by point c, cannot influence the change in the parameter n. If $c \in G_2$, the change in n may, generally speaking, cause D° to vanish. Then to study the rotation stability of the corresponding gyrostat we must retain in the expansion of H terms of order higher than the fourth. If c belongs to the resonance hypersurface, a change in n can, generally speaking, give rise to a change in the stability of the corresponding rotation to instability, and vice versa.

A study of the stability of the uniform rotations (2.1) of a gyrostat on an absolutely smooth horizontal plane was carried out when in a small neighbourhood of the contact point the surface of the housing is specified by Eqs.(3.1) and (3.2). We note that using expansion (2.2) of Hamilton's function, one can study the stability of a gyrostat with a surface different from (3.1) or (3.2). Here the coefficient of the Hamiltonian in formulae (3.4) and (3.5) changes but formulae (5.5)–(5 10) and (5.13) still hold.

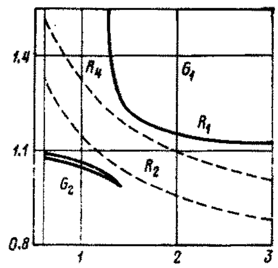
6. The sufficient conditions of stability of the uniform rotations (2.1) of a gyrostat were obtained in terms of a constraint of the inequality type imposed on the coefficients of the normal forms of the Hamiltonian. These equations have a cumbersome form and, therefore, to verify that the corresponding inequalities are satisfied a computer was used. Since the number of parameters was high, for clear interpretations of the results obtained it is necessary to consider special cases. Below we list briefly the results of a study of two such cases.

Let us assume that $a = 3/2$; then the domain F has the form

$$F = \{c\colon\, b \in (3/5, 3),\; l_1 > 0,\; l_2 > 0\}$$

We choose $\varepsilon = \pi/4$, $r_1/h_1 = 1/2$, $r_2/h = 7/10$, then $l = 1/10$, $l_1 = l_2 = 3/5$ and we set $n = 1$.

6.1. Let $\omega_2 = 0$. This means that the gyroscope does not rotate in relation to the housing, i.e. it is an absolute solid. The figure shows domains G_1 and G_2. Domain G_1 has a lower bound set by a branch of the curve $R_1\,(Q_2 = 0)$ which has a vertical asymptote $b = 1$, and for $\omega_1 = 1.126$ it intersects the straight line $b = 3$. The uniform rotations of the solid which correspond to R_1 are stable. The size of the domain G_2 is small compared with that of G_1, and all of it is placed in a rectangle $b \in (0.6;\ 1.389)$, $\omega_1 \in (0.985;\ 1.084)$. The domain G_2 has an upper bound set by a



branch of curve R_1 and a lower bound set by a branch of the curve $R_2\,(Q_1^2 - 4Q_2 = 0)$. The uniform rotations (2.1) corresponding to R_1 are not stable, unlike the solutions of (2.1) corresponding to R_2. The curves R_4 and $\{D^\circ = 0\}$ do not intersect the domain G_2 and are not shown in the figure. We note that the boundaries of G_1 and G_2 have no common point since $l = {}^1/_{10} \neq 0$ (for the problems discussed in /12–14/, it is the point where both coefficients of Poincaré stability vanish).

6.2. Let $\omega_1 = 0$. This means that the housing is in equilibrium, and the gyroscope continues to rotate. In this case the domain G_1 is empty. The domain G_2 has a lower bound set by a branch of curve R_2 (shown by a dashed line in the figure), which for $\omega_2 = 1.323$ intersects the straight line $b = {}^3/_5$ and for $\omega_2 = 0.880$ the straight line $b = 3$. Curve R_4 is situated slightly above curve R_2 (curve R_4 is also shown by a dashed line in the figure). The uniform rotations (2.1) which correspond to these curves are stable. The curve $\{D^\circ = 0\}$ does not intersect domain G_2.

The domains of stability G_1 and G_2 are constructed for the case when $\omega_1, \omega_2 > 0$. For $\omega_1, \omega_2 < 0$ the corresponding domains are symmetrical about those constructed with respect to the Ob axis.

After analysing the special cases we may conclude that the rotation mode discussed in Sect.6.2 is prefereable in view of the stability of the rotation mode described in Sect.6.1.

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